

# Relevant boundary perturbations of CFT: A case study

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## ABSTRACT

We consider simple CFT models which contain massless bosons, massless fermions or a supersymmetric combination of the two, on the strip. We study the deformations of these models by relevant boundary operators. In particular, we work out the details for a boundary operator with a quadratic dependence on the fields and argue that some of our results can be extended to a more general situation. In the fermionic models, several subtleties arise due to a doubling of zero modes at the UV fixed point and a “GSO projected” RG flow. We attempt to resolve these issues and to discuss how bulk symmetries are realised along the flow. We end with some speculations on possible string theory applications of these results.

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# 1 Introduction and motivation

Boundary conformal field theories (BCFT) find applications in vastly different branches of physics ranging from condensed matter systems to string theory.

BCFTs can be perturbed by an operator  $\Phi$  which couples only to the boundary. Such a perturbation leads to new boundary conditions which may even break conformal invariance. If the operator  $\Phi$  has conformal dimension  $h = 1$ , it is possible to stay at the conformal point to all orders in the coupling constant. However if  $h \neq 1$ , the boundary perturbation introduces a length scale into the theory and there is hence a RG-flow. When  $h > 1$  the perturbation is irrelevant and the flow keeps us within the same BCFT; for  $h < 1$  the perturbation is relevant and the BCFT we flow to in the IR is different to the one we started out with. In general it is a difficult problem to find the IR BCFT corresponding to a given relevant perturbation, nevertheless, several non-trivial examples have been worked out using the thermodynamic Bethe ansatz [1] and perturbative techniques [2]. The study of relevant boundary perturbations of BCFTs is mainly motivated by quantum impurity problems in condensed matter theory [3] and by tachyonic condensation in string theory [4].

In the present note we would like to study relevant boundary perturbations of a few simple BCFTs. In contrast to previous works, which were based on integrable model or perturbative techniques, we use the canonical quantization approach and concentrate on possible algebraic structures that may arise. We consider several free theories which are solvable and can be regarded as a reasonable first approximation to interacting theories. We argue that some results obtained for free theories can be extended to general interacting theories as well.

Our main motivation for this work comes from string theory applications (i.e., the RNS string model in a tachyonic background field [4], [5]). However we hope that some of our results might also applied to other branches of physics such as impurity problems. Therefore, throughout the paper we try to avoid any direct references to string theory, reserving such comments/discussions for the end.

We define our theory, which is conformally invariant in the bulk, on a strip  $R \times [0, L]$  with two spatial boundaries. In general, both boundaries can have different boundary interactions. Though we work with Minkowski signature, there are no problems in principle, in going to the Euclidean version and considering the same kinds of theories on a disk, an annulus and etc. In all cases, this generalization is straightforward and throughout the paper, we comment on the Euclidean versions of the models we consider.

The paper is organized as follows: In section 2 we consider a theory with a massless free boson living in the bulk and a potential coupled to the boundary. We argue that for a

general potential, the chiral and anti-chiral algebras survive, and infact become non-trivially related to each other through the boundary perturbation. We work out these algebras in detail, assuming a quadratic form for the boundary perturbation. In section 3 we consider the fermionic couterpart of the previous model. It turns out that the construction of relevant boundary perturbations for these models involves some subtleties due to appearance of extra zero modes. Some general statements about such 'extra zero modes' have previously been made in the string theory literature, [6] and [4]. We try, in the framework of our model, to present a detailed discussion of the mechanism whereby these zero modes arise.

Using the above results, we construct a supersymetric model in section 4. We are able to do this only for a quadratic boundary potential and we offer some reasons as why it might be difficult to do so in general. In section 5 we go back to the bosonic and fermionic models but this time, with different perturbations on each of the two boundaries; we discuss the realization of bulk symmetries in this case. In section 6 we speculate on the string theory applications of our results. The construction of the RNS string model in a tachyonic background field seems to be quite restrictive and we have been able handle only a quadratic tachyonic field. This could be a sign that possible tachyonic backgrounds are limited by self-consistency of the theory, but ofcourse, more work needs to be done before one can reach any final conclusions. We also propose directions for further study.

## 2 Free boson theory

In this section we study a simple BCFT perturbed by a relevant boundary operator. We consider the theory on a strip with the following Lagrangian

$$L = \frac{1}{2} \int_0^L d\sigma \partial_\alpha \phi \partial^\alpha \phi + \lambda V(\phi(0)) - \lambda V(\phi(L)), \quad (2.1)$$

and Minkowski signature  $(1, -1)$ . The Lagrangian (2.1) gives rise to the following equation of motion and boundary conditions

$$(\partial_\tau^2 - \partial_\sigma^2)\phi = 0, \quad (\phi' + \lambda \partial_\phi V(\phi))|_{0,L} = 0. \quad (2.2)$$

The boundary conditions have the same form on both boundaries, however the model is not parity invariant. A parity transformation  $\Omega$  ( $\sigma \rightarrow L - \sigma$ ) is in fact equivalent to changing the sign of the coupling constant  $\lambda$ . The equation of motion (2.2) is satisfied if  $\phi(\tau, \sigma)$  is the sum of two arbitrary functions of  $(\tau + \sigma)$  and  $(\tau - \sigma)$ .

Introducing the notation  $z = \exp(-i\frac{\pi}{L}(\tau + \sigma))$  and  $\bar{z} = \exp(-i\frac{\pi}{L}(\tau - \sigma))$  we can define

the following currents

$$J(z) = \sqrt{\frac{2}{L}} \partial_+ \phi = \sum_n J_n z^n, \quad \bar{J}(\bar{z}) = \sqrt{\frac{2}{L}} \partial_- \phi = \sum_n \bar{J}_n \bar{z}^n. \quad (2.3)$$

where  $\partial_{\pm} = \partial_{\tau} \pm \partial_{\sigma}$ . The Wick rotation  $\tau \rightarrow it$  makes  $J(z)$  holomorphic and  $\bar{J}(\bar{z})$  anti-holomorphic. Canonical commutation relations imply that these currents satisfy the following chiral and antichiral algebras

$$[J_n, J_m] = n\delta_{n+m}, \quad [\bar{J}_n, \bar{J}_m] = n\delta_{n+m}. \quad (2.4)$$

The boundary conditions in (2.2) lead to a nontrivial relation between  $J_n$  and  $\bar{J}_n$ . However, this relation should be such that it respects the algebra (2.4). In other words, holomorphic and anti-holomorphic currents are linked by a unitary transformation in the corresponding Fock space.

Using the Sugawara construction we can write down generators

$$L_k = \frac{1}{2} \sum_n J_n J_{k-n}, \quad \bar{L}_k = \frac{1}{2} \sum_n \bar{J}_n \bar{J}_{k-n} \quad (2.5)$$

which obey the standard Virasoro algebras. Note that  $L_k \neq \bar{L}_k$ . This is as it should be, since the Lagrangian (2.1) is not invariant under conformal transformations.

We now want to work out a simple example, using a specific form for the boundary perturbation. Let us take  $V(\phi) = \frac{1}{2}\phi^2$  which makes the boundary conditions linear

$$(\phi' + \lambda\phi)|_{0,L} = 0. \quad (2.6)$$

Equation (2.2) subject to boundary conditions (2.6) can be reduced to a Sturm-Liouville problem. The properly normalized solution is given by

$$\phi(\tau, \sigma) = \frac{\sqrt{2L}}{\pi} \sum_{n \neq 0} \frac{e^{-in\frac{\pi}{L}\tau}}{\sqrt{n^2 + \tilde{\lambda}^2}} a_n [\cos(n\frac{\pi}{L}\sigma) - \frac{\tilde{\lambda}}{n} \sin(n\frac{\pi}{L}\sigma)] \quad (2.7)$$

where  $\tilde{\lambda} = \frac{L\lambda}{\pi}$  is the dimensionless coupling constant. At  $\lambda = 0$ , due to the zero modes that arise, the above solution contains an extra term,  $a_0\tau$ . Using completeness of eigenfunctions of the corresponding Sturm-Liouville problem, one can find the commutation relations  $[a_n, a_m] = n\delta_{n+m}$ . Reality of  $\phi$  implies that  $a_n^+ = a_{-n}$ .

The components of chiral and antichiral currents are now given by

$$J_n = -\frac{\tilde{\lambda} + in}{\sqrt{n^2 + \tilde{\lambda}^2}} a_n, \quad \bar{J}_n = \frac{\tilde{\lambda} - in}{\sqrt{n^2 + \tilde{\lambda}^2}} a_n. \quad (2.8)$$

and are related by the following transformation

$$J_n - e^{i\varphi(n, \tilde{\lambda})} \bar{J}_n = 0, \quad e^{i\varphi(n, \tilde{\lambda})} = \frac{in + \tilde{\lambda}}{in - \tilde{\lambda}}. \quad (2.9)$$

Altogether (2.4) and (2.9) imply the following algebra

$$[J_n, J_m] = n\delta_{n+m}, \quad [\bar{J}_n, \bar{J}_m] = n\delta_{n+m}, \quad [J_n, \bar{J}_m] = e^{i\varphi(n, \tilde{\lambda})} n\delta_{n+m}. \quad (2.10)$$

The above algebra is all we need to calculate Green's functions of chiral and antichiral currents (or other objects constructed from them). From the algebra we can see that the correlators

$$\langle J(z_1)J(z_2)\dots J(z_k) \rangle, \quad \langle \bar{J}(\bar{z}_1)\bar{J}(\bar{z}_2)\dots \bar{J}(\bar{z}_k) \rangle \quad (2.11)$$

are exactly the same as in the unperturbed conformal field theory. However the mixed correlators

$$\langle J(z_1)J(z_2)\dots J(z_p)\bar{J}(\bar{z}_{p+1})\bar{J}(\bar{z}_{p+2})\dots \bar{J}(\bar{z}_k) \rangle \quad (2.12)$$

are different because of the last commutation relation in (2.10). It is easy to convince oneself that this statement is rather generic for relevant boundary perturbations, though the explicit form of the third commutation relation may vary. Certainly, the relation between  $J_n$  and  $\bar{J}_n$  will not always be linear. In general, a representation of the Virasoro algebra is labelled by the eigenvalue of the  $a_0$  operator. However, along the flow, there is no  $J_0$  (i.e no zero mode  $a_0$ ), and hence there is one unique representation.

Using the definitions (2.5) one can obtain left and right Virasoro generators that obey the required Virasoro algebras independently. However  $L_k \neq \bar{L}_k$  except in the limiting values of the coupling constant,  $\tilde{\lambda} = 0$  and  $\tilde{\lambda} = \infty$ . As can be seen from (2.9)

$$\begin{aligned} \lim_{\tilde{\lambda} \rightarrow 0} e^{i\varphi(n, \tilde{\lambda})} = 1 & \Rightarrow J_n - \bar{J}_n = 0 \quad \text{at} \quad \tilde{\lambda} = 0, \\ \lim_{\tilde{\lambda} \rightarrow \infty} e^{i\varphi(n, \tilde{\lambda})} = -1 & \Rightarrow J_n + \bar{J}_n = 0 \quad \text{at} \quad \tilde{\lambda} = \infty. \end{aligned} \quad (2.13)$$

The above relations between  $J_n$  and  $\bar{J}_n$  imply that we start out with Neumann boundary conditions in the UV and flow to Dirichlet conditions in the IR.

### 3 Free fermion theory

We would now like to construct the fermionic counterpart of the previous example. Since we want to eventually combine bosonic and fermionic theories to form a single supersymmetric model, we simply obtain the fermionic theory by supersymmetrising the bosonic one.

Using the supersymmetric transformation

$$\delta\phi = -\epsilon^+\psi_+ - \epsilon^-\psi_-, \quad (3.14)$$

on equation (2.2), we get the fermionic boundary conditions

$$[\partial_\sigma(\eta\psi_+ + \psi_-) + \lambda\partial_\phi^2 V(\phi)(\eta\psi_+ + \psi_-)]|_{0,L} = 0, \quad (3.15)$$

where  $\epsilon^+ = \eta\epsilon^-$ , for  $\eta = \pm 1$ , and  $\psi_\pm$  are components of a Majorana spinor. The relative sign of  $\eta$  determines whether we are in the Neveu-Schwarz ( $\eta_0 = -\eta_L$ ) or Ramond ( $\eta_0 = \eta_L$ ) sector.

In analogy to the bosonic case, we would expect that the boundary condition (3.15) interpolates between Neumann and Dirichlet conditions at  $\lambda = 0$  and  $\lambda = \infty$  respectively. This is almost, but not quite, true. The complication arises due to having a dimensionful coupling constant  $\lambda$  in the boundary condition (3.15); to make the dimensions come out right, we must also include a derivative on the fermions. Thus, in the UV limit ( $\lambda = 0$ ) we actually have  $\partial_\sigma(\eta\psi_+ + \psi_-) = 0$  rather than the standard Neumann condition  $\eta\psi_+ - \psi_- = 0$ . For nonzero modes these conditions are completely equivalent. However subtleties arise when we consider the zero modes. Our 'modified' Neumann condition contains a derivative and thus does not impose any constraints on the zero modes unlike the standard Neumann case which reduces the number of zero modes from two to one.

The boundary condition (3.15) then, can be seen to describe a flow along which the number of zero modes (or alternately, the degeneracy of the ground state) decreases. Also, the UV fixed point of this flow does not coincide with that of the standard fermionic model precisely because of this 'doubling' of zero modes.

We now analyze the fermionic model in detail. Since we are perturbing only by a boundary operator, the bulk physics remains unchanged and we have the usual equations of motion

$$\partial_+\psi_- = 0, \quad \partial_-\psi_+ = 0. \quad (3.16)$$

Taking into account (3.16) the conditions (3.15) can be rewritten as follows

$$[\partial_\tau(\eta\psi_+ - \psi_-) + \lambda\partial_\phi^2 V(\phi)(\eta\psi_+ + \psi_-)]|_{0,L} = 0. \quad (3.17)$$

For the sake of simplicity we take  $V(\phi) = \frac{1}{2}\phi^2$ .

We now construct the action which gives rise to the above equations of motions and boundary conditions. Since in writing down the action we would like to avoid having extra derivatives on fermions, we introduce an auxiliary fermion  $d$  on the boundary, as follows

$$[\partial_\tau d + \sqrt{\lambda}(\eta\psi_+ + \psi_-)]|_{0,L} = 0, \quad [\sqrt{\lambda}d - (\eta\psi_+ - \psi_-)]|_{0,L} = 0 \quad (3.18)$$

where  $d$  can, in general, be different on the two boundaries.

The action

$$S = i \int d^2\sigma [\psi_+ \partial_- \psi_+ + \psi_- \partial_+ \psi_-] + \frac{i}{2} \int d\tau [d \partial_\tau d + \sqrt{\lambda} d (\eta \psi_+ + \psi_-)]|_0^L \quad (3.19)$$

reproduces the required equations of motion (3.16) and boundary conditions (3.18). This action is suitable for the  $\lambda = 0$  limit. An equivalent action can be obtained for the  $\lambda = \infty$  limit by absorbing  $\sqrt{\lambda}$  into the definition of  $d$ . As far as we know, an auxiliary boundary fermion has also been used in the study of the Kondo model with a bulk mass term [7].

It is evident from the action (3.19) (or, alternatively, from the boundary conditions (3.18)) that in the UV limit ( $\lambda = 0$ ), the boundary fermion  $d$  is needed only for extra zero modes and does not play any other role. Such a boundary fermion was also introduced by Witten in [6], for slightly different reasons.

As we flow to the IR ( $\lambda \rightarrow \infty$ ),  $d$  decouples completely. We would like to emphasize that this boundary fermion  $d$  is just a useful tool which facilitates analysis of the theory. All the results we obtain using  $d$  could equally well have been obtained from a direct analysis of the condition (3.15).

The model given by the action (3.19) can be solved explicitly. We start by considering the Ramond sector, where the mode expansion is over integers. For  $\lambda \neq 0$ , the properly normalized solutions of (3.16) and (3.18) are

$$\psi_+(\tau + \sigma) = \frac{1}{\sqrt{L}} \sum_{r \in \mathbb{Z}} \frac{ir + \tilde{\lambda}}{\sqrt{r^2 + \tilde{\lambda}^2}} \theta_r e^{-ir \frac{\pi}{L}(\tau + \sigma)}, \quad \psi_-(\tau - \sigma) = \frac{1}{\sqrt{L}} \sum_{r \in \mathbb{Z}} \eta \frac{ir - \tilde{\lambda}}{\sqrt{r^2 + \tilde{\lambda}^2}} \theta_r e^{-ir \frac{\pi}{L}(\tau - \sigma)} \quad (3.20)$$

where the modes obey standard anticommutation relations  $\{\theta_r, \theta_s\} = \delta_{r+s}$ .

We can now introduce the fermionic currents

$$j(z) = \sqrt{L} \psi_+ = \sum_{r \in \mathbb{Z}} j_r z^r, \quad \bar{j}(\bar{z}) = \sqrt{L} \psi_- = \sum_{r \in \mathbb{Z}} \bar{j}_r \bar{z}^r \quad (3.21)$$

where  $z$  and  $\bar{z}$  have been defined previously. Our boundary conditions result in the following relation between components of currents

$$j_r - \eta e^{i\varphi(r, \tilde{\lambda})} \bar{j}_r = 0, \quad e^{i\varphi(r, \tilde{\lambda})} = \frac{ir + \tilde{\lambda}}{ir - \tilde{\lambda}}, \quad (3.22)$$

where the phase  $e^{i\varphi(r, \tilde{\lambda})}$  has the same limits as in (2.13). This, together with the anticommutation relation  $\{j_r, j_s\} = \delta_{r+s}$ , leads to the following algebra

$$\{j_r, j_s\} = \delta_{r+s}, \quad \{\bar{j}_r, \bar{j}_s\} = \delta_{r+s}, \quad \{j_r, \bar{j}_s\} = \eta e^{i\varphi(s, \tilde{\lambda})} \delta_{r+s}. \quad (3.23)$$

Using (3.23) one can construct generators which obey the left and right Virasoro algebras. These are given by:

$$L_k = \frac{1}{2} \sum_r (r + \frac{1}{2}k) j_{-r} j_{k+r}, \quad \bar{L}_k = \frac{1}{2} \sum_r (r + \frac{1}{2}k) \bar{j}_{-r} \bar{j}_{k+r}. \quad (3.24)$$

Note that  $L_k \neq \bar{L}_k$ . For  $\lambda \neq 0$  there is just one zero mode  $\theta_0$  (hence there is just one ground state)<sup>3</sup>.

At the UV fixed point however, there are two zero modes  $\theta_0^+$  and  $\theta_0^-$ . The canonical commutation relations between  $\psi_+$  and  $\psi_-$  imply that the zero modes obey  $\{\theta_0^\alpha, \theta_0^\beta\} = \delta^{\alpha\beta}$ . The ground state is labelled by a representation of the corresponding two-dimensional Clifford algebra. We can define  $(-1)^F$  in a natural way, as follows:

$$(-1)^F = \theta_0^+ \theta_0^- (-1)^N, \quad N = \sum_r \theta_{-r} \theta_r. \quad (3.25)$$

The two vacua then have opposite  $(-1)^F$  eigenvalues. World-sheet parity,  $\Omega$ , plays a remarkably similar role. On the modes, it acts as follows:

$$\begin{aligned} \Omega \theta_0^+ \Omega^{-1} &= \theta_0^- \\ \Omega \theta_0^- \Omega^{-1} &= \theta_0^+ \\ \Omega \theta_r \Omega^{-1} &= e^{i\pi r} \theta_r \end{aligned} \quad (3.26)$$

where we have used  $\Omega \psi_+ \Omega^{-1} = \psi_-$  and  $\Omega \psi_- \Omega^{-1} = \psi_+$ .

Since  $\Omega$  interchanges  $\theta_0^+$  and  $\theta_0^-$ , the two vacua have opposite eigenvalues under parity. There is ofcourse an overall ambiguity as to which of the two vacuums we choose to be odd and which to be even under parity. We can fix this freedom by requiring that each of the two vacua should have identical eigenvalues under both  $\Omega$  and  $(-1)^F$ . By interchanging the two zero modes, parity also changes the value of  $(-1)^F$ .

It is rather interesting to note that since  $\Omega$  and  $(-1)^F$  perform identical functions, we can also obtain the standard Ramond sector by moding out the Ramond sector in our theory, by worldsheet parity  $\Omega$ . Thus, at  $\lambda = 0$ , making a GSO projection is 'equivalent' to orientifolding the model by  $\Omega$ .

From (3.15), it is clear that for  $\lambda \neq 0$ , a worldsheet parity transformation is not a symmetry and is infact equivalent to a change in the sign of  $\lambda$ . At the level of the action one can see this by rescaling  $d$  by  $\sqrt{\lambda}$

$$S = i \int d^2\sigma [\psi_+ \partial_- \psi_+ + \psi_- \partial_+ \psi_-] + \frac{i}{2} \int d\tau [\frac{1}{\lambda} d \partial_\tau d + d(\eta \psi_+ + \psi_-)]|_0^L. \quad (3.27)$$

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<sup>3</sup>This statement applies to Majorana spinors. For Dirac spinors, one zero mode gives rise to a twice degenerate vacuum [8], because of the algebra  $\{\theta_0, \theta_0^\dagger\} = 1$ , where  $\dagger$  denotes Dirac conjugation.



At the UV fixed point,  $\Omega : \sigma \rightarrow L - \sigma$  takes  $\lambda$  to  $-\lambda$  and  $d$  to  $-d$ . Due to the boundary fermions (or two fold degeneracy of the ground state), the standard Ramond sector (with only one vacuum) can be thought of as the GSO projected Ramond sector, as obtained at the UV fixed point, of our model. This picture is similar to what Witten has proposed in [6]. When  $\lambda$  becomes non zero the model picks up one of the two ground states and flows to the IR. In this sense one can say that the RG flow is 'GSO projected' with respect to the UV fixed point. Which particular ground state is projected out depends on the sign of  $\lambda$ .

The NS sector can also be treated in a similar fashion. For  $\lambda \neq 0$ , the equations (3.20)-(3.23) carry through to the NS sector, with the modification that  $r$  now takes half integer values. However, things get a little more complicated at the UV point fixed point. Once again (as in the R sector), the vacuum has a two fold degeneracy. This degeneracy can be attributed to the two extra boundary fermions that exist at the two boundaries. Since these fermions do not enter into the Hamiltonian, they do not affect the energy of the system; all they do is provide a mechanism for making the vacuum degenerate. In this sector, we define the action of parity<sup>4</sup> as follows:  $\Omega\psi_+\Omega^{-1} = -\psi_-$  and  $\Omega\psi_-\Omega^{-1} = \psi_+$ . This implies that

$$\begin{aligned}\Omega\theta_0^+\Omega^{-1} &= -\theta_0^- \\ \Omega\theta_0^-\Omega^{-1} &= \theta_0^+ \\ \Omega\theta_r\Omega^{-1} &= e^{i\pi r}\theta_r\end{aligned}\tag{3.28}$$

With  $\Omega$  defined thus, and  $(-1)^F$  defined as in (3.25), we can now proceed just as we did in the Ramond sector.

So far our arguments were based on the canonical quantization approach. We were trying to quantize the theory using the bulk equations of motion (3.16) and the boundary conditions (3.15), so the action (3.19) was useful but was not necessary. However the action becomes a crucial tool when one makes a Wick rotation and goes to the Euclidean version; for (3.19), this has the following form

$$S = i \int_{\Sigma} d^2z [\psi \partial_{\bar{z}} \psi + \bar{\psi} \partial_z \bar{\psi}] - \frac{i}{2} \int_{\partial\Sigma} [d\partial_t d + \sqrt{\lambda} d(\psi + \bar{\psi})] \tag{3.29}$$

where  $\Sigma$  is some domain in  $R^2$  and  $t$  parameterizes the boundary of  $\Sigma$ . Starting from (3.19) we made the Wick rotation ( $\tau = it$ ) and redefined the spinors  $\psi = z^{1/2}\psi_+$  and  $\bar{\psi} = \bar{z}^{1/2}\psi_-$  (from now on  $\psi_+$  and  $\psi_-$  should be thought of as Weyl spinors which are complex conjugates

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<sup>4</sup> At the level of the action, we have the freedom of defining worldsheet parity  $\Omega$  such that  $\Omega^2 = 1$  or  $-1$ . We decide which alternative is more suitable by appealing to the boundary conditions. In the R sector, we find that boundary conditions are preserved if we choose  $\Omega^2 = 1$  while in the NS sector, boundary conditions are parity invariant when parity is defined such that  $\Omega^2 = -1$  [9].

of each other).  $d$  is a real boundary fermion (equivalently, it can be made purely imaginary). Fermions are periodic or antiperiodic depending on whether they belong to the R or NS sector. This model can now be treated using standard path integral techniques, where integration over  $d$  is assumed.

## 4 Supersymmetric theory

Based on results from the two previous sections we can now construct the supersymmetric version of our theory. The supersymmetric action is simply the sum of (2.1) and (3.19)

$$S = \frac{1}{2} \int d^2\sigma [\partial_- \phi \partial_+ \phi + i\psi_+ \partial_- \psi_+ + i\psi_- \partial_+ \psi_-] - \frac{1}{2} \int d\tau \lambda \phi^2|_0^L + \frac{i}{4} \int d\tau [d \partial_\tau d + \sqrt{\lambda} d (\eta \psi_+ + \psi_-)]|_0^L. \quad (4.30)$$

This action gives rise to the boundary conditions (2.6) and (3.18). The presence of a boundary introduces subtleties with supersymmetry, hence, together with bulk supersymmetry transformations, we need to use the above boundary conditions as well, when checking for supersymmetry [10]. Using the bulk transformations

$$\delta\phi = -\epsilon^+ \psi_+ - \epsilon^- \psi_-, \quad \delta\psi_+ = -i\epsilon^+ \partial_+ \phi, \quad \delta\psi_- = -i\epsilon^- \partial_- \phi, \quad (4.31)$$

together with the transformation of the boundary fermion

$$\delta d = 2i\epsilon \sqrt{\lambda} \phi, \quad (4.32)$$

one can show that the supersymmetric variation of the action (4.30) is zero, modulo boundary conditions (2.6) and (3.18). This action is similar to the component form action proposed in [4] for the case of a quadratic tachyonic potential. However it seems that in general, the proposed action does not have the right properties. The superfield formalism does not guarantee supersymmetry in the presence of a boundary, unless boundary conditions are suitably taken into account, in accordance with procedures outlines in [10].

So far we have been unable to construct the supersymmetric action for a general potential  $V(\phi)$ . However, the general boundary conditions (2.2) and (3.17) are compatible with supersymmetry and ensure closure of the supersymmetric algebra classically.

The supersymmetric model with a quadratic potential can be worked out explicitly. Starting from the Ramond sector, using the solutions (2.7) and (3.20) we can define the components of the currents  $J_n$  and  $j_r$  as in (2.8) and in (3.21) correspondingly. We can also construct generators

$$L_k = \frac{1}{2} \sum_n J_n J_{k-n} + \frac{1}{2} \sum_r (r + \frac{1}{2}k) j_{-r} j_{k+r}, \quad G_k = \sum_n J_{-n} j_{k+n} \quad (4.33)$$

which obey the super Virasoro algebra. In the same way, generators  $\bar{L}_k$  and  $\bar{G}_k$  (which obey the same algebra) can be constructed using the antiholomorphic currents  $\bar{J}_n$  and  $\bar{j}_n$ . For  $\tilde{\lambda} \neq 0$  one can see that  $L_k \neq \bar{L}_k$  and  $G_k \neq \bar{G}_k$ . This is as it should be, as super conformal invariance is broken by boundary interactions. At the IR fixed point ( $\tilde{\lambda} = \infty$ ), the left and right super Virasoro algebras coincide. At the UV fixed point we have the extended super Virasoro algebra [11] (i.e., super Virasoro algebra together with  $(-1)^F$ ).

An identical analysis can be performed for the NS sector with the only difference that now we have half integer moding for fermionic currents  $j_r, \bar{j}_r$  and generators  $G_k, \bar{G}_k$ . At the UV fixed point, due to the doubly degenerate vacuum, we would have two copies of the representation of NS super Virasoro algebra.

One may wonder about the need of introducing a degeneracy into the vacuum at the UV fixed point. There is a simple explanation. Suppose that in the action (4.30) we replace the bosonic boundary potential by the following expression

$$-\frac{1}{2}\lambda \int d\tau (\phi - \phi_1)^2|_L + \frac{1}{2}\lambda \int d\tau (\phi - \phi_0)^2|_0. \quad (4.34)$$

The action thus modified will still be supersymmetric. Depending on the sign of  $\lambda$ , we will flow, in the IR to different theories. One of these will have  $\phi|_0 = \phi_0, \quad \phi|_L = \phi_1$  and the other,  $\phi|_0 = \phi_1, \quad \phi|_L = \phi_0$ . These theories are related by a parity transformation  $\Omega$  and they each have their own vacuum. Therefore, in order to be able to get either of these theories in the IR, we need to have both the corresponding vacua present in the theory at the UV fixed point.

## 5 Non-trivial realization of bulk symmetries

In this section we would like to investigate the possibility of having different perturbations on the two boundaries. We start by considering the bosonic model given by the following action

$$L = \frac{1}{2} \int_0^L d\sigma \partial_\alpha \phi \partial^\alpha \phi + \lambda V(\phi(0)) - \gamma V(\phi(L)). \quad (5.35)$$

This model has a rich phase structure with four fixed points NN, ND, DN and DD, where N and D stand for Neumann and Dirichlet boundary conditions respectively. Obviously, our previous bosonic model can be easily embedded into this new one. Also, when  $\gamma = -\lambda$ , the model becomes parity invariant, unlike previous examples.

As before, we can work out the quadratic potential  $V(\phi) = \frac{1}{2}\phi^2$  in detail. In this case we

get the following boundary conditions

$$(\phi' + \lambda\phi)|_0 = 0, \quad (\phi' + \gamma\phi)|_L = 0. \quad (5.36)$$

The solution of the massless Klein-Gordon equation with boundary conditions (5.36) can also be reduced to a Sturm-Liouville problem, the normalized solution of which is given by the following expression

$$\phi(\tau, \sigma) = \frac{\sqrt{2L}}{\pi} \sum_{\tilde{n} \neq 0} \frac{e^{-i\tilde{n}\frac{\pi}{L}\tau}}{\sqrt{\tilde{n}^2 + \tilde{\lambda}^2}} a_{\tilde{n}} [\cos(\tilde{n}\frac{\pi}{L}\sigma) - \frac{\tilde{\lambda}}{\tilde{n}} \sin(\tilde{n}\frac{\pi}{L}\sigma)], \quad (5.37)$$

where  $\tilde{n}$  is subject to the transcendental equation

$$\tan(\tilde{n}\pi) = \frac{(\tilde{\gamma} - \tilde{\lambda})\tilde{n}}{\tilde{n}^2 + \tilde{\gamma}\tilde{\lambda}} \quad (5.38)$$

with  $\tilde{\lambda} = \frac{\lambda L}{\pi}$  and  $\tilde{\gamma} = \frac{\gamma L}{\pi}$ . Equation (5.38) cannot be solved explicitly, but positive solutions  $\tilde{n}(\tilde{\lambda}, \tilde{\gamma})$  can be ordered and presented in the following form

$$\tilde{n}(\tilde{\lambda}, \tilde{\gamma}) = n + g(\tilde{\lambda}, \tilde{\gamma}, n), \quad n \in Z_+, \quad 0 \leq g(\tilde{\lambda}, \tilde{\gamma}, n) < 1 \quad (5.39)$$

where  $g$  is a function of  $\tilde{\lambda}$ ,  $\tilde{\gamma}$  and  $n$  which can be computed numerically. Negative solutions are obtained trivially by taking  $\tilde{n} \rightarrow -\tilde{n}$ . The asymptotic behaviour of  $g$  is clear; it must be zero at the fixed points  $(\tilde{\lambda} = 0, \tilde{\gamma} = 0)$ ,  $(\tilde{\lambda} = \infty, \tilde{\gamma} = \infty)$  and  $1/2$  at the fixed points  $(\tilde{\lambda} = \infty, \tilde{\gamma} = 0)$ ,  $(\tilde{\lambda} = 0, \tilde{\gamma} = \infty)$ .

We can now relabel  $a_{\tilde{n}}$  as  $a_n$  and define chiral and antichiral currents which have the following components

$$J_n = -\frac{\tilde{\lambda} + i\tilde{n}}{\sqrt{\tilde{n}^2 + \tilde{\lambda}^2}} e^{-i\frac{\pi}{L}(\tau+\sigma)g(\tilde{\lambda}, \tilde{\gamma}, n)} a_n, \quad \bar{J}_n = \frac{\tilde{\lambda} - i\tilde{n}}{\sqrt{\tilde{n}^2 + \tilde{\lambda}^2}} e^{-i\frac{\pi}{L}(\tau-\sigma)g(\tilde{\lambda}, \tilde{\gamma}, n)} a_n. \quad (5.40)$$

Thus the components of chiral and antichiral currents are related as follows:

$$J_n - e^{i\varphi(n, \tilde{\lambda}, \tilde{\gamma})} \bar{J}_n = 0, \quad e^{i\varphi(n, \tilde{\lambda}, \tilde{\gamma})} = \frac{i\tilde{n} + \tilde{\lambda}}{i\tilde{n} - \tilde{\lambda}} e^{-2i\frac{\pi}{L}\sigma g(\tilde{\lambda}, \tilde{\gamma}, n)}. \quad (5.41)$$

Using canonical commutation relations, together with the fact that eigenfunctions of a Sturm-Liouville problem form a complete set, one can show that the currents (5.40) obey the same algebra as (2.10) but with a new spatial dependent phase  $e^{i\varphi(n, \tilde{\lambda}, \tilde{\gamma})}$ .

As before, using the Sugawara construction (2.7) we construct left and right Virasoro generators  $L_n$  and  $\bar{L}_n$  such that  $L_n \neq \bar{L}_n$  at a generic point in the  $(\tilde{\lambda}, \tilde{\gamma})$ -plane. At the fixed points  $(\tilde{\lambda} = 0, \infty; \tilde{\gamma} = 0, \infty)$  conformal symmetry is restored and we have  $L_n = \bar{L}_n$ . In the

present model we see that bulk conformal symmetry is realised rather non-trivially. The representation of the left Virasoro algebra has a natural realization on the Fock space built up by  $J_n$  whereas the representation of the right Virasoro algebra would be realized naturally on the Fock space built up by  $\bar{J}_n$ . These two Fock spaces are related through a spatially dependent unitary transformation (5.41). Hence,  $\sigma$ -independence for  $L_n$  in a particular basis would necessarily imply  $\sigma$  dependence for  $\bar{L}_n$ .

In general, the model (5.35) is not parity invariant. A parity transformation  $\Omega$  amounts to changing the signs of the coupling constants:  $(\lambda, \gamma) \rightarrow (-\lambda, -\gamma)$ . However, there is a fixed line  $\gamma = -\lambda$  under this transformation where the model becomes parity invariant. For a model defined on this line, parity symmetry would be realized in a highly nontrivial fashion since its action on the Fock space would be  $\sigma$ -dependent. At the self-dual point  $\sigma = L/2$ , the left and right Virasoro generators coincide  $L_n = \bar{L}_n$ . It is not clear to us how to interpret this result.

The fermionic model can also be easily generalized in the same fashion. The most general scenario for fermions is given by the following boundary conditions

$$\begin{aligned} [\partial_\tau(\eta_0\psi_+ - \psi_-) + \lambda(\eta_0\psi_+ + \psi_-)]|_0 &= 0 \\ [\partial_\tau(\eta_L\psi_+ - \psi_-) + \gamma(\eta_L\psi_+ + \psi_-)]|_L &= 0 \end{aligned} \quad (5.42)$$

where  $(\eta_0, \eta_L)$  corresponds to the choice of spin structure at the two boundaries. The general solutions to this problem are given below

$$\psi_+(\tau+\sigma) = \frac{1}{\sqrt{L}} \sum_{\tilde{r}} \frac{i\tilde{r} + \tilde{\lambda}}{\sqrt{\tilde{r}^2 + \tilde{\lambda}^2}} \theta_{\tilde{r}} e^{-i\tilde{r}\frac{\tau}{L}(\tau+\sigma)}, \quad \psi_-(\tau-\sigma) = \frac{1}{\sqrt{L}} \sum_{\tilde{r}} \eta_0 \frac{i\tilde{r} - \tilde{\lambda}}{\sqrt{\tilde{r}^2 + \tilde{\lambda}^2}} \theta_{\tilde{r}} e^{-i\tilde{r}\frac{\tau}{L}(\tau-\sigma)} \quad (5.43)$$

where  $\tilde{r}$  is subject to the constraints

$$\tan(\tilde{r}\pi) = \frac{(\tilde{\gamma} - \tilde{\lambda})\tilde{r}}{\tilde{r}^2 + \tilde{\gamma}\tilde{\lambda}}, \quad \text{for } \eta_0\eta_L = 1, \quad (5.44)$$

$$\tan(\tilde{r}\pi) = \frac{\tilde{r}^2 + \tilde{\gamma}\tilde{\lambda}}{(\tilde{\gamma} - \tilde{\lambda})\tilde{r}}, \quad \text{for } \eta_0\eta_L = -1. \quad (5.45)$$

Thus, depending on the sector,  $\tilde{r}$  obeys different transcendental equations. In the sector  $\eta_0\eta_L = 1$  the condition (5.44) for fermions coincides with the bosonic condition (5.38), so fermionic currents  $j_r$  and  $\bar{j}_r$  can be constructed just as in the bosonic case. These currents are related as follows:

$$j_r - e^{i\varphi(n, \tilde{\lambda}, \tilde{\gamma})} \bar{j}_r = 0, \quad (5.46)$$

where the spatial dependent phase is given by second equation in (5.41). It is fairly straightforward to construct the left and right super Virasoro algebras. Their realizations on the Fock space are  $\sigma$ -dependent, exactly as for the bosonic model.

For the sector  $\eta_0\eta_L = -1$ , fermionic and bosonic modings are different. This is not surprising since, even at the fixed points, the moding is different for bosons and NS fermions. We work around this problem by using the same tricks as in the bosonic case. It is natural now to write  $\tilde{r}$  as the sum of a function and a half integer; this makes the  $\sigma$ -dependent phase between  $j_r$  and  $\bar{j}_r$  different to what it was before (5.41). The left and right super Virasoro algebras can be constructed in the usual way.

So far we have considered the theory away from critical points. As before, the UV fixed point needs special consideration due to the extra zero modes which arise there. Once again, there is a natural  $(-1)^F$ -operator which defines the GSO projection and the RG flow can be thought of as 'GSO projected' from the UV point of view, since the degeneracy of the ground state is lifted. In the present model, parity does not in general play the role it played in section 3. Away from the line  $\lambda = -\gamma$  we can use parity the way we used it earlier and argue that the change  $(\lambda, \gamma) \rightarrow (-\lambda, -\gamma)$  leads to a change in the value of  $(-1)^F$ ; at the line  $\lambda = -\gamma$  where parity is realized non trivially, we cannot use this argument.

## 6 String theory applications

The main string theory applications of our results come from the boundary renormalization group interpretation of open string tachyon condensation [4]. Formally one can rewrite the action (4.30) for a number of bosons and fermions, labelled by an external space time index:  $X^\mu = \sqrt{2\pi\alpha'}\phi^\mu$  and  $\Psi_\pm^\mu = \sqrt{2\pi\alpha'}\psi_\pm^\mu$ , where  $X$  and  $\Psi$  are canonically normalized. The boundary potential, called the tachyon profile henceforth, can now depend on more than one direction. Various string models arise, resulting from the freedom of assigning different boundary conditions. The analysis of such models should be similar to what we have outlined in this paper for a more simple case. Even without studying these models in detail however, we can find certain restrictions they need to obey.

Depending on the details of the string configuration we choose and the number of directions on which the tachyonic profile depends, we get an even or odd dimensional Clifford algebra for zero modes at the UV fixed point. It is obvious that the operator  $(-1)^F$  can be defined meaningfully only for an even dimensional Clifford algebra [6]. Thus, it is only these cases to which we can hope our general arguments will apply. This may be interpreted as a restriction on the D-brane systems which can consistently couple to a particular tachyonic profile.

Another problem is how to properly interpret the UV fixed point of this model within the framework of string theory i.e, finding a string theory interpretation of the auxiliary fermions

living on the boundary. These boundary fermions increase the degeneracy of the ground state, having much the same effect as Chan-Paton factors. Keeping this analogy in mind, we suppose that the UV fixed point should correspond not to a single standard open string sector but instead to a few sectors, the particulars of which are determined by the algebra of the boundary fermions.

We would also like to point out the following fact: In order to couple an open string to a tachyonic background, we need to introduce boundary fermions and through them, a number of degenerate open string vacua. Building on the analogy with Chan-Paton factors, it would seem that these open string vacua interpolate between different D-brane configurations; i.e., that it is not consistent to couple a background tachyon to an open string living on a single D-brane.

There are several obvious directions for further work. To begin with, it would be worthwhile to write down the supersymmetric action for a general tachyonic background; we have been able to do so only in the case when the tachyonic profile is quadratic. Also, while it is clear that our arguments can only be expected to work for even dimensional Clifford algebras, explicit examples of such cases have not yet been looked at. Concrete configurations of D-branes can be studied to work out the nature of the restrictions on the tachyonic profiles which may consistently couple to it, and indeed to verify that such restrictions even exist. Similarly, the correspondence between the degeneracy introduced by Chan-Paton factors and that introduced by boundary fermions should be explored in more detail and the relation made precise. This, again, should tell us exactly which configurations of D-branes need to be present in order for us to couple the open strings to a tachyonic profile. Of course we would expect the allowed D-brane configurations resulting from both the above analyses to be identical.

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## References

- [1] P. Fendley, H. Saleur and N. P. Warner, “Exact solution of a massless scalar field with a relevant boundary interaction,” Nucl. Phys. **B430** (1994) 577 [hep-th/9406125];  
P. Dorey, A. Pocklington, R. Tateo and G. Watts, “TBA and TCSA with boundaries and excited states,” Nucl. Phys. **B525** (1998) 641 [hep-th/9712197];

- F. Lesage, H. Saleur and P. Simonetti, “Boundary flows in minimal models,” *Phys. Lett.* **B427** (1998) 85 [hep-th/9802061].
- [2] P. Fendley, F. Lesage and H. Saleur, “Solving 1-d plasmas and 2-d boundary problems using Jack polynomials and functional relations,” *J. Stat. Phys.* **79** (1995) 799 [hep-th/9409176];  
A. Recknagel, D. Roggenkamp and V. Schomerus, “On relevant boundary perturbations of unitary minimal models,” hep-th/0003110.
- [3] H. Saleur, “Lectures on non perturbative field theory and quantum impurity problems,” cond-mat/9812110; “Lectures on non perturbative field theory and quantum impurity problems. II,” cond-mat/0007309.  
I. Affleck, “Conformal Field Theory Approach to the Kondo Effect,” *Acta Phys. Polon.* **B26** (1995) 1869 [cond-mat/9512099].
- [4] J. A. Harvey, D. Kutasov and E. J. Martinec, “On the relevance of tachyons,” hep-th/0003101.
- [5] T. Z. Husain and M. Zabzine, “Bosonic open strings in a tachyonic background field,” hep-th/0005202.
- [6] E. Witten, “D-branes and K-theory,” *JHEP* **9812** (1998) 019 [hep-th/9810188].
- [7] Z. S. Bassi and A. LeClair, “The Kondo Model with a Bulk Mass Term,” *Nucl. Phys.* **B552** (1999) 643 [hep-th/9811138].
- [8] P. Ginsparg, “Applied Conformal Field Theory,” HUTP-88-A054 *Lectures given at Les Houches Summer School in Theoretical Physics, Les Houches, France, Jun 28 - Aug 5, 1988*.
- [9] E. G. Gimon, J. Polchinski “Consistency Conditions for Orientifolds and D-Manifolds,” *Phys. Rev.* **D54** (1996) 1667 [hep-th/9601038].
- [10] P. Haggi-Mani, U. Lindström and M. Zabzine, “Boundary conditions, supersymmetry and A-field coupling for an open string in a B-field background,” *Phys. Lett.* **B483** (2000) 443 [hep-th/0004061].
- [11] J. D. Cohn and D. Friedan, “Super Characters And Chiral Asymmetry In Superconformal Field Theory,” *Nucl. Phys.* **B296** (1988) 779.